Coding for Euclidean Space:

Past, Present and Challenges Ahead

Gottfried Ungerboeck

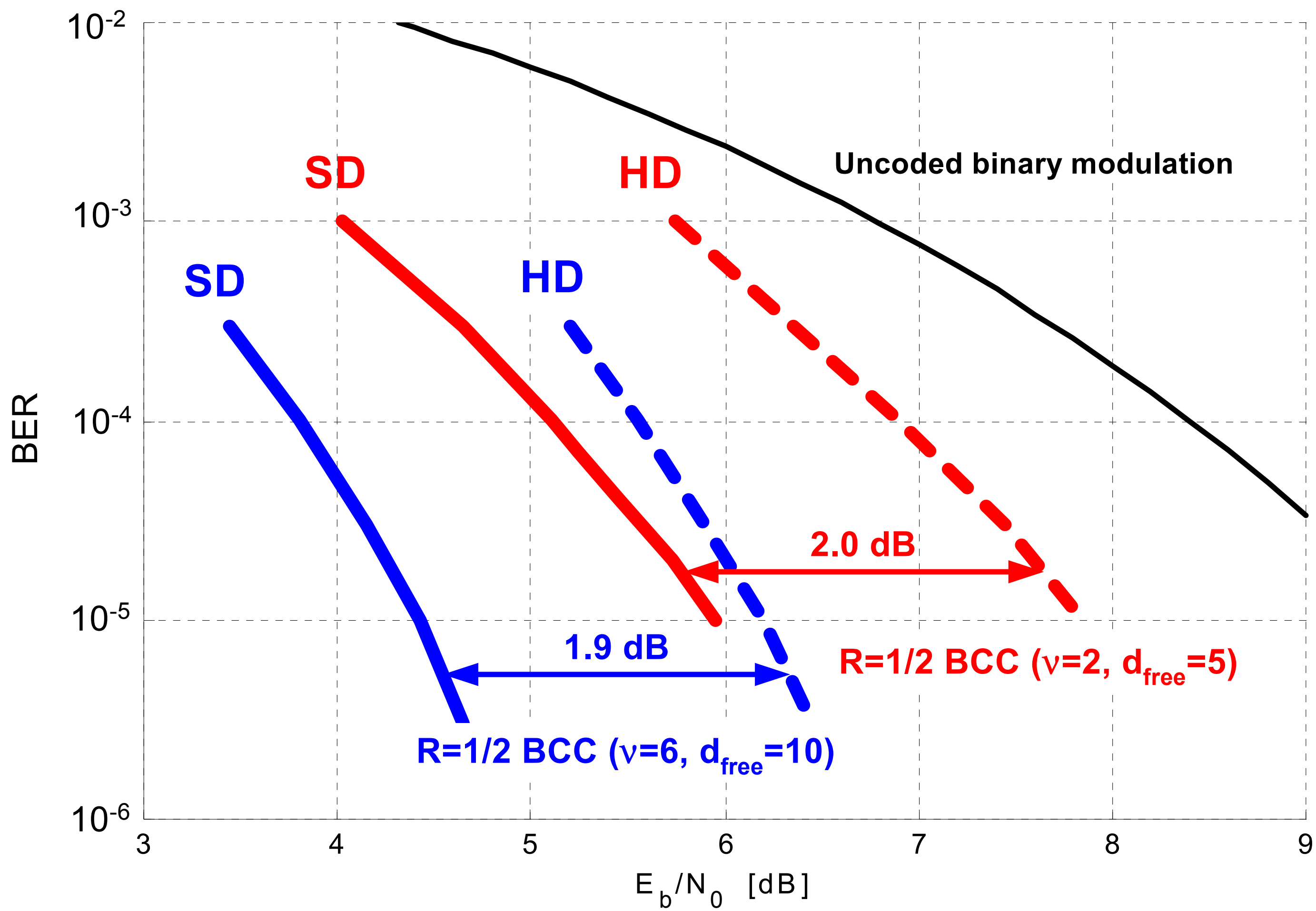
Broadcom Corporation

- **Past:** **from soft-decoding of binary convolutional codes to trellis coded modulation, multilevel coding, dense lattices.**

- **Present:** **concatenated coding and iterative decoding, joint equalization and decoding, MIMO systems, space-time coding.**

- **Outlook:** **what comes next – another big topic? extended phase of consolidations? What are the challenges?**

Soft / hard decoding (SD/HD) of binary convolutional codes

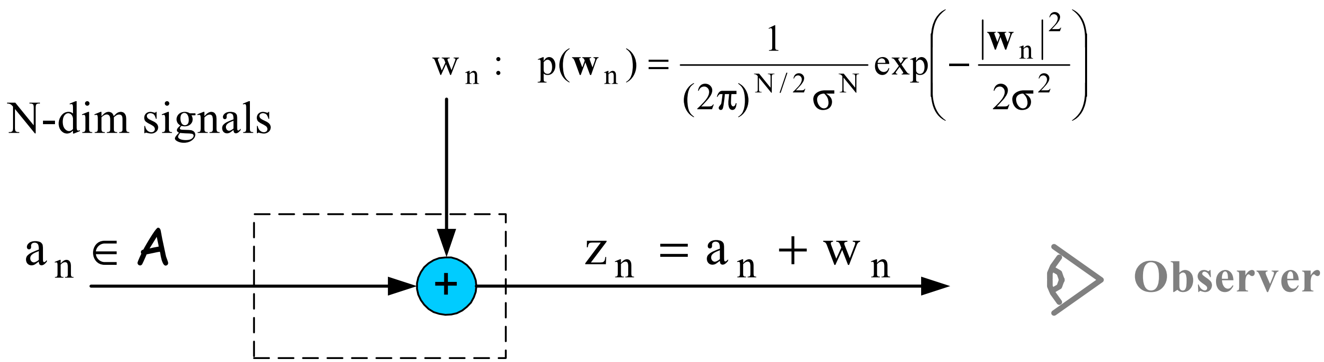


Heller and Jacobs, 1971 / Lin & Costello, 1st ed, 1983

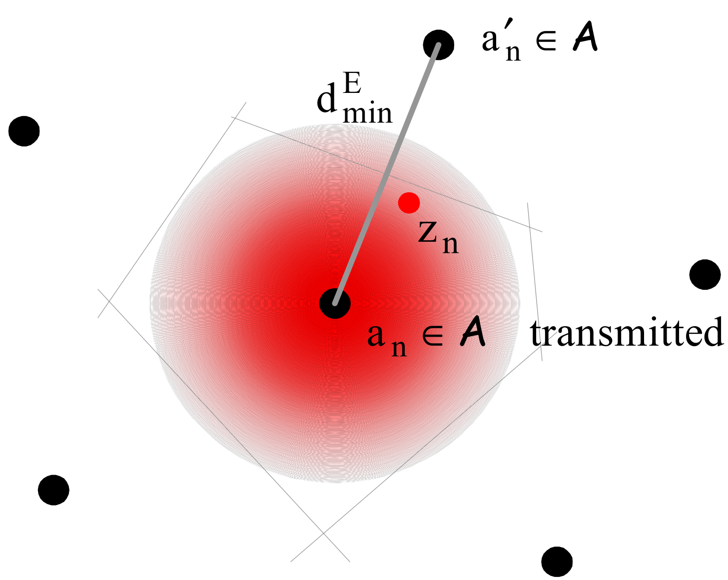
1965 - 1975

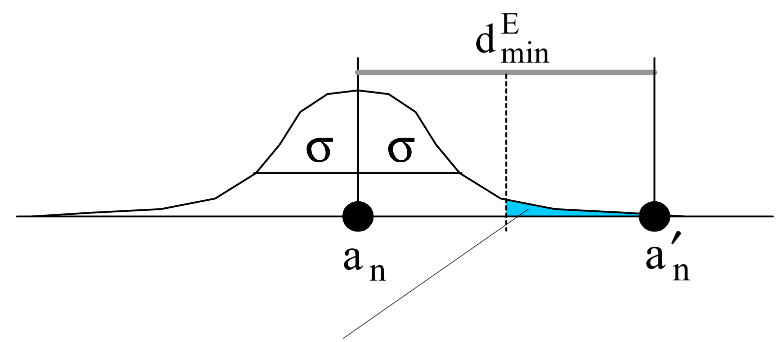
- Known: soft decoding of binary codes can provide an improvement of ≈ 2 dB over hard decoding.
- Designing signal codes with large free Euclidean distance could have easily been seen as an important goal.
- But research concentrated on "error control codes", usually in combination with binary modulation, where Hamming and Euclidean distance are equivalent.
- The paradigm was: hard-decision symbol decoders can make errors; so, transmit redundant check bits and let an error control decoder detect and/or correct the errors.

Receiving signals in additive white Gaussian noise



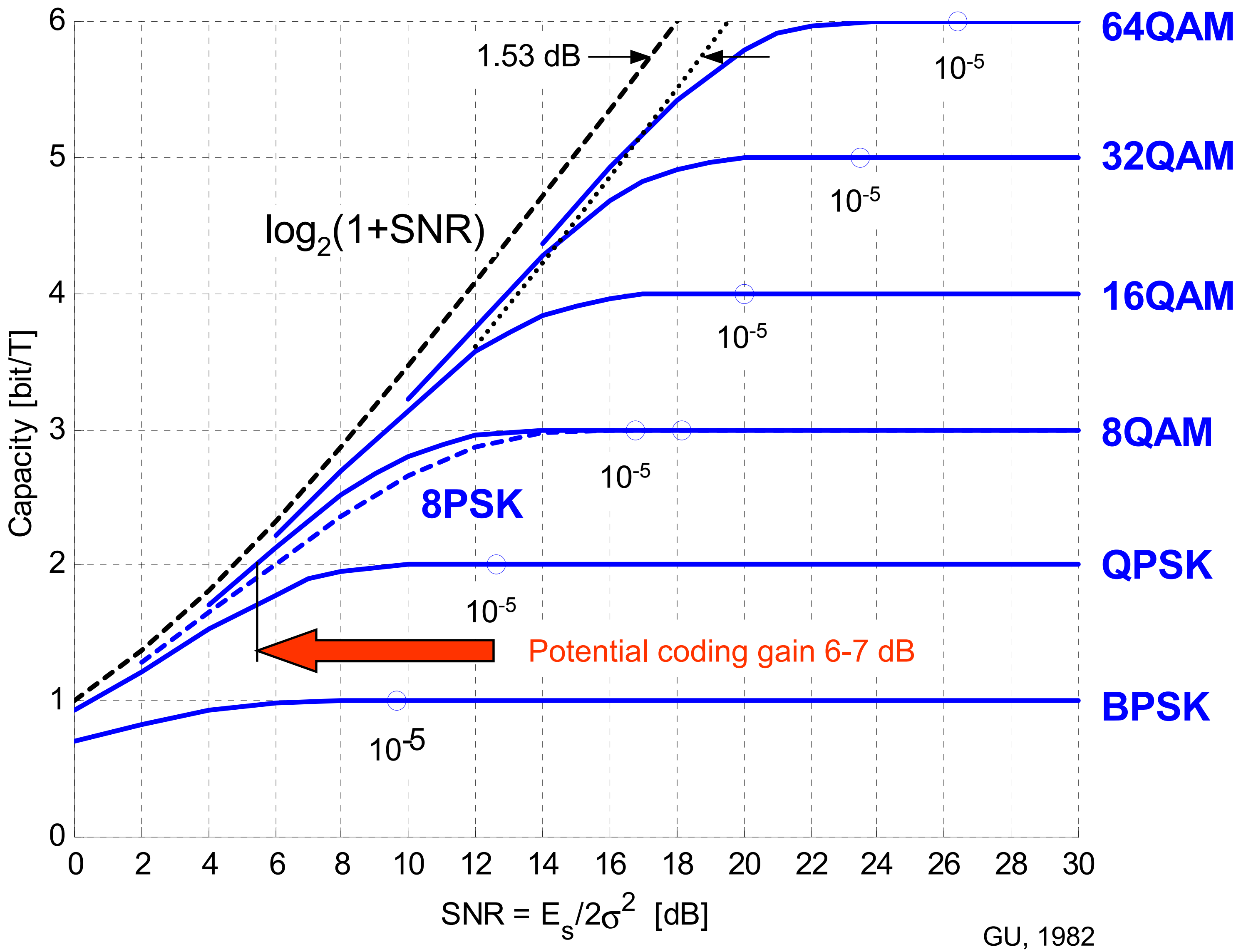
$$\text{ML decoder} \quad \hat{\mathbf{a}}_n = \arg \operatorname*{Min}_{\boldsymbol{\alpha} \in A} |\mathbf{z}_n - \boldsymbol{\alpha}|^2$$





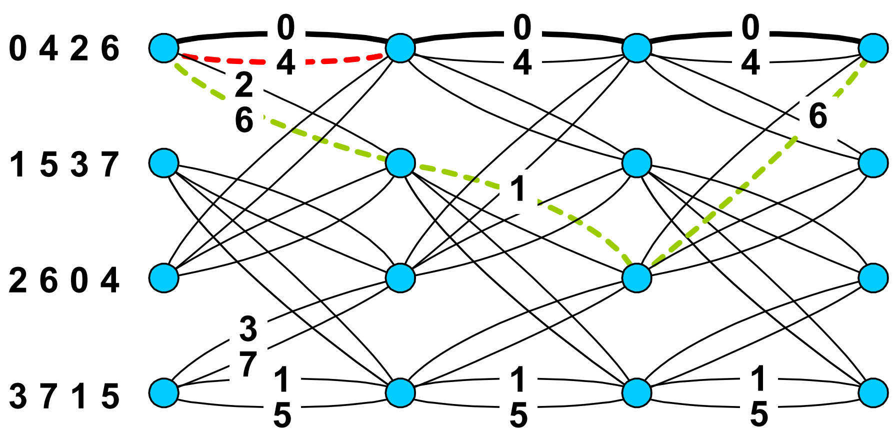
$$\Pr[\mathbf{a}'_n \text{ more likely than } \mathbf{a}_n / \mathbf{z}_n] = Q\left(\frac{d^E_{min}}{2\sigma}\right)$$

Capacity of 2-D AWGN channel with discrete input / soft output

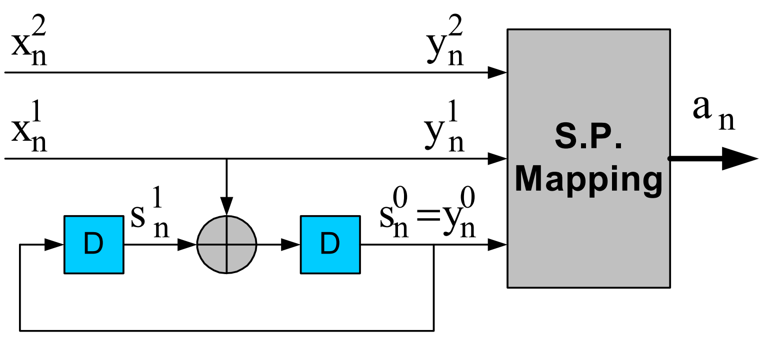


Trellis coded modulation: the first useful code (1975)

4-state coded 8PSK: 3 dB gain over uncoded QPSK (same rate and bandwidth)

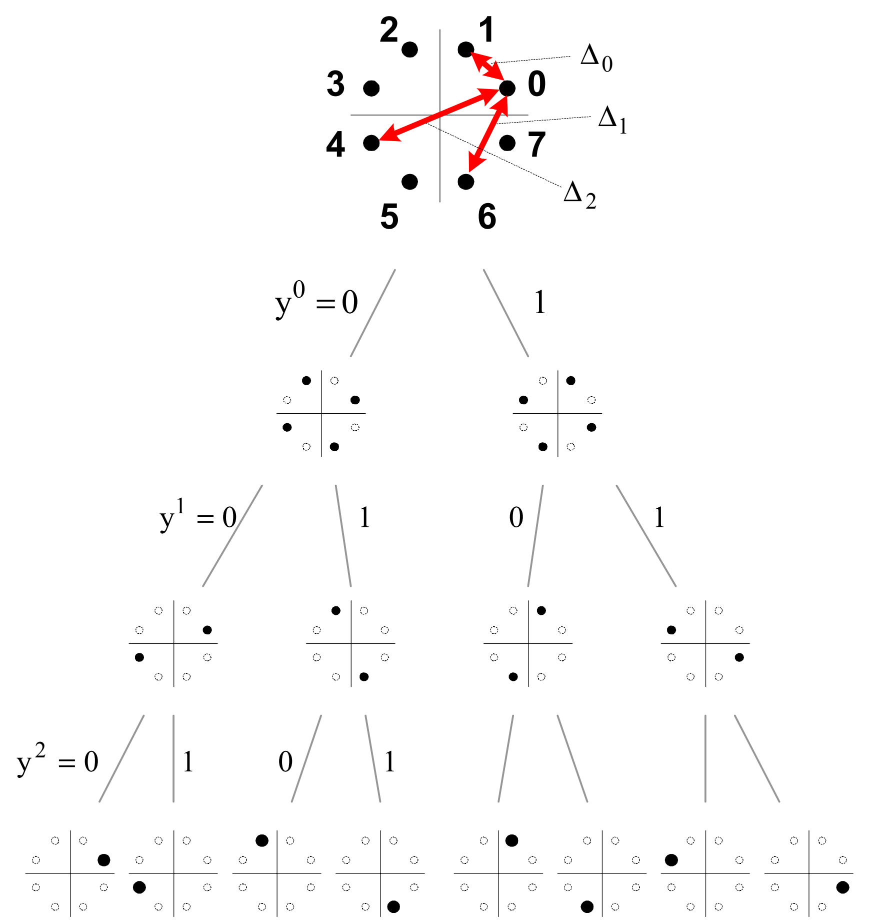


Four-state trellis diagram

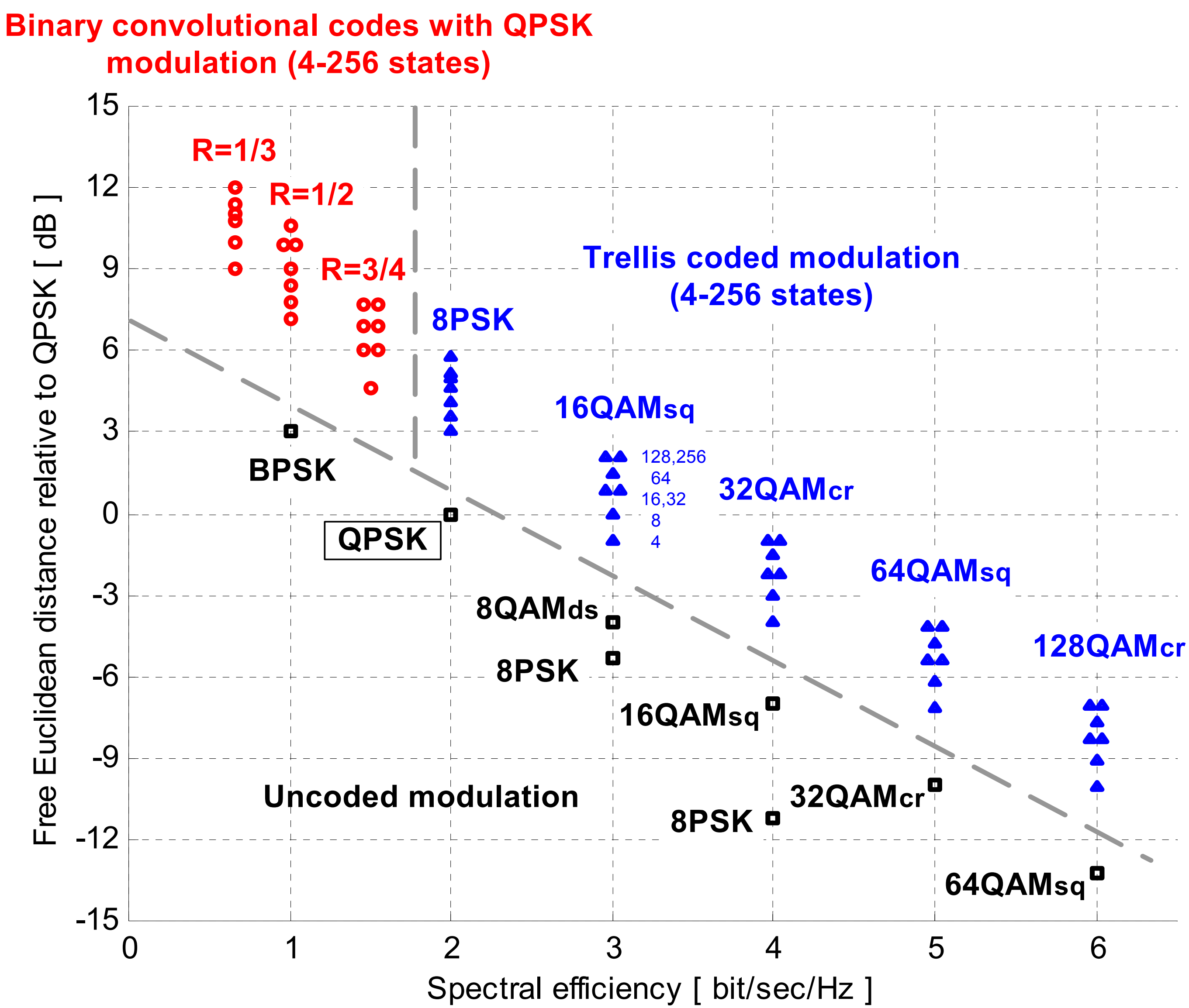


Four-state systematic recursive encoder

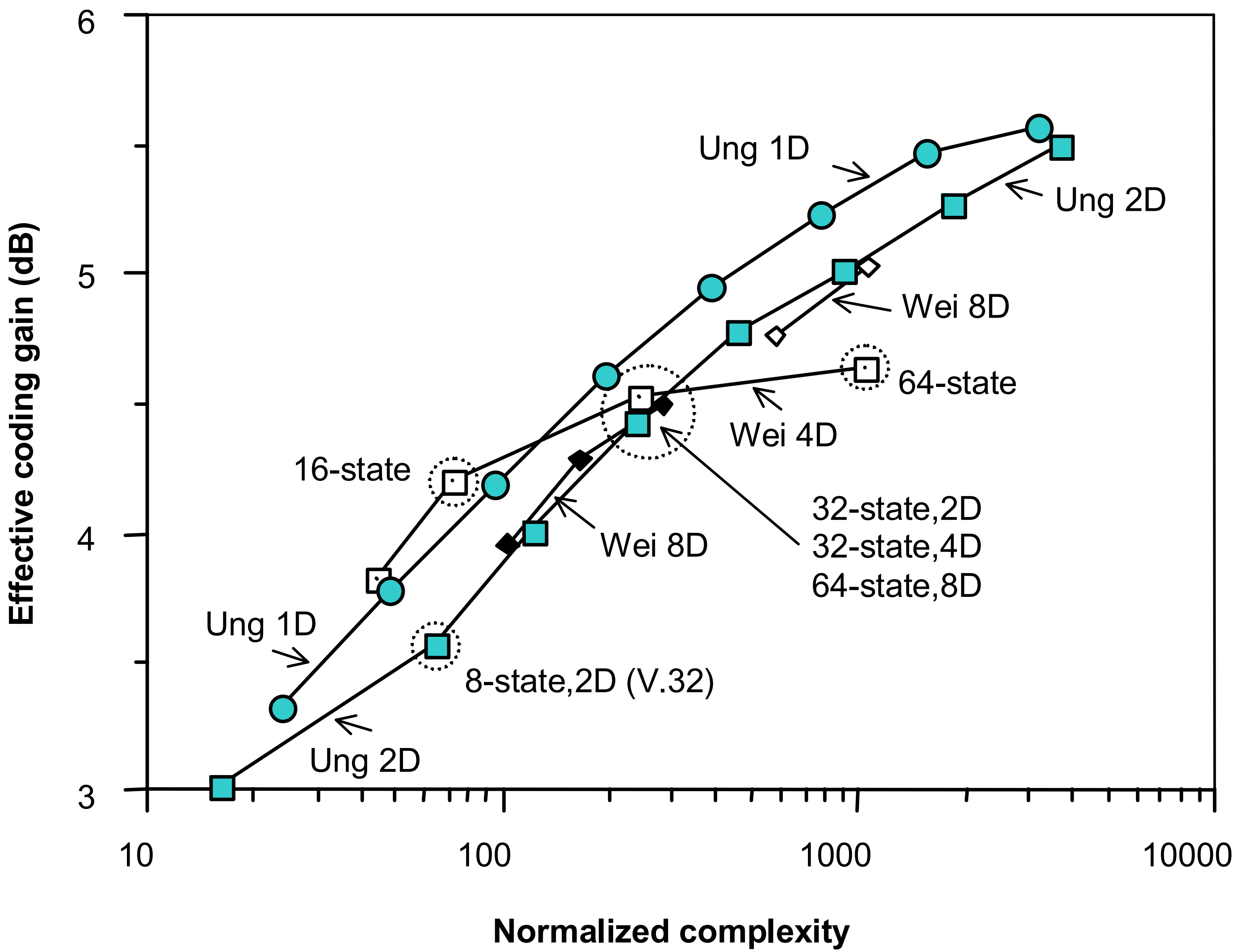
Set partitioning / mapping



From binary modulation to higher order modulations



TCM: Effective coding gain vs. decoding complexity

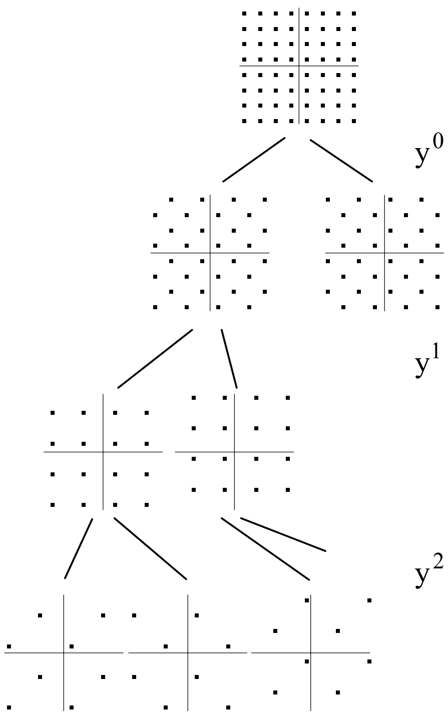


Applications of TCM

- Dial-up modems: V.32, V.17, V.34, V.90, V.92
- Digital subscriber links: SHDLS, ADSL, VDSL
- Cable modems: downstream J.83, upstream Docsis 2.0
- Terrestrial TV: VSB modem
- Ethernet: 802.3 1 Gbit/s over copper
- Wireless LAN: 802.11g
- Mobile telephony: GSM Edge
- WPAN: 802.15 … etc.

Multilevel coding and multistage decoding (Imai&Hirakawa 1977)

(Lattice = infinite symbol constellation with algebraic group properties)



$\mathbf{Z}^2 : \Delta_0^2$

y^0

$R\mathbf{Z}^2 : \Delta_1^2 = 2\Delta_0^2$

y^1

$2\mathbf{Z}^2 : \Delta_2^2 = 4\Delta_0^2$

y^2

$2R\mathbf{Z}^2 : \Delta_3^2 = 8\Delta_0^2$

Multi-level construction: Gosset lattice E_8 from $(Z^2)^4$

Z^2	Z^2	Z^2	Z^2	(n, k, d^H)
c_1^0	c_2^0	c_3^0	c_4^0	$\in \mathcal{C}^0 = \text{RM}(2,0) = (4, 1, 4)$
c_1^1	c_2^1	c_3^1	c_4^1	$\in \mathcal{C}^1 = \text{RM}(2,1) = (4, 3, 2)$
c_1^2	c_2^2	c_3^2	c_4^2	$\in \mathcal{C}^2 = \text{RM}(2,2) = (4, 4, 1)$

$$\left(d_{\min}^E\right)^2 = \text{Min}\left[\Delta_0^2 \times 4,\ \Delta_1^2 \times 2,\ \Delta_2^2 \times 1\right] = 4\Delta_0^2$$

$$\text{Coding gain } E_8 \text{ over } \mathbf{Z}^8 : \ \gamma_c = \frac{\left(d_{\min}^E\right)^2}{\Delta_0^2} \times \left(\frac{2^{1+3+4}}{2^{12}}\right)^{\frac{2}{8}} = 2 \ \ (3\text{ dB})$$

Multistage decoding (= suboptimal bounded distance decoding)

1. Find most-likely codeword $\hat{\mathbf{c}}^0 \in \mathcal{C}^0$, assuming unconstrained $\mathbf{c}^1, \mathbf{c}^2$
2. Find most-likely codeword $\hat{\mathbf{c}}^1 \in \mathcal{C}^1$ for given $\mathbf{c}^0 = \hat{\mathbf{c}}^0$, assuming unconstrained $\mathbf{c}^2$
3. Find most-likely codeword $\hat{\mathbf{c}}^2 \in \mathcal{C}^2$ for given $\mathbf{c}^0 = \hat{\mathbf{c}}^0$ and $\mathbf{c}^1 = \hat{\mathbf{c}}^1$.

Dense lattices

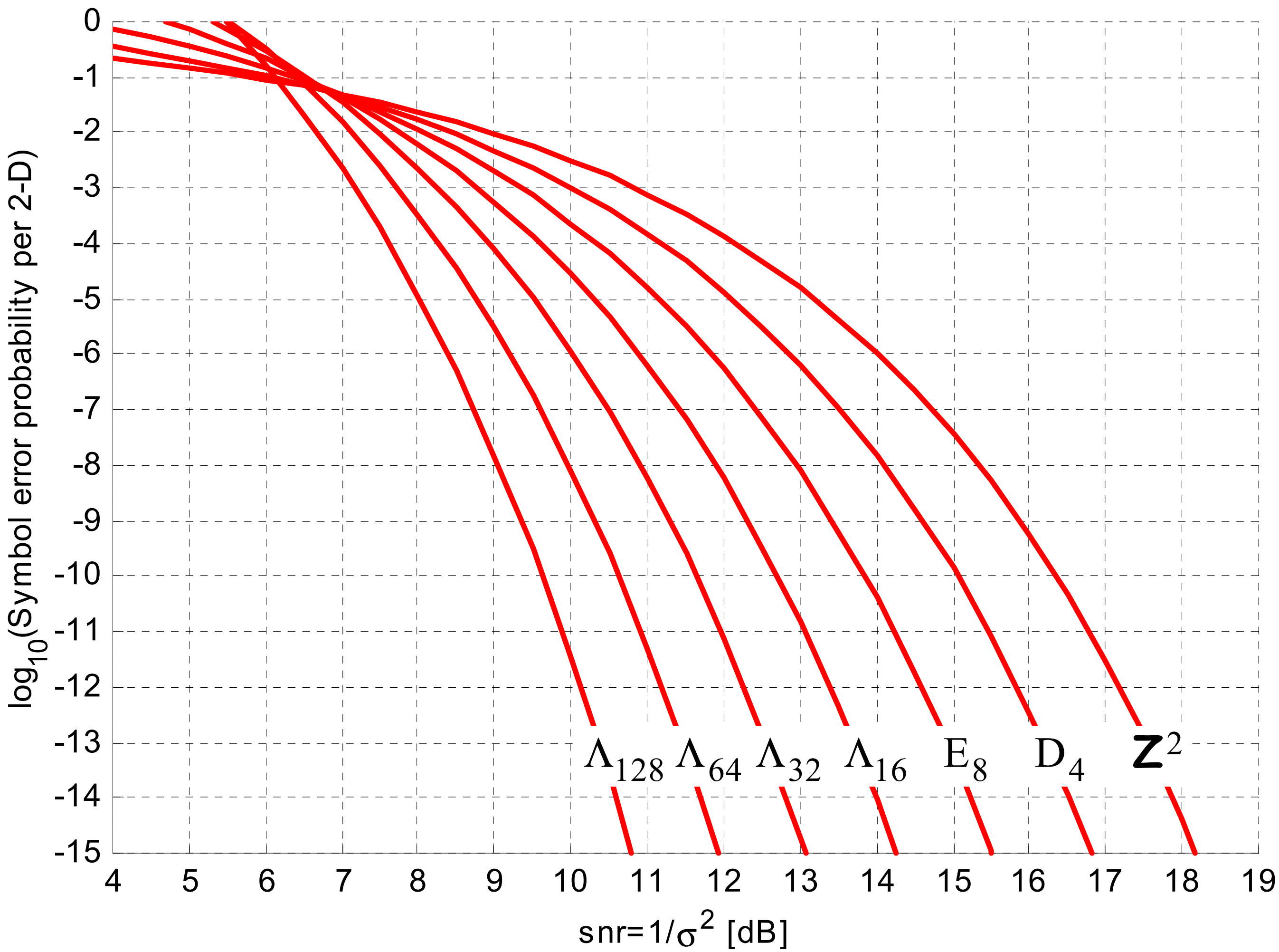
Barnes-Wall Lattices

	$\mathbf{Z}^2$	Schläfi D_4	Gosset E_8	Λ_{16}	Λ_{32}	Λ_{64}	Λ_{128}	$\cdots$
$\gamma_c =$	1 (0 dB)	$\sqrt{2}$ (1.5 dB)	2 (3 dB)	$\sqrt{2}$ (1.5 dB)	4 (6 dB)	$4\sqrt{2}$ (7.5 dB)	8 (9 dB)	$\cdots$
$K_{min} =$	4	24	240	4'320	146'880	9'694'080	1'260'230'400	$\cdots$
$K_{min} / N =$	2	6	30	270	4590	151'470	9'845'550	$\cdots$

Constructions are based on set partitioning of component integer lattices and multilevel coding with Reed-Muller codes (many equivalent constructions)

	Hexagonal H_2	Leech L_{24}
$\gamma_c =$	1.15 (0.62 dB)	4 (6 dB)
$K_{min} =$	6	196'560
$K_{min} / N =$	3	8190

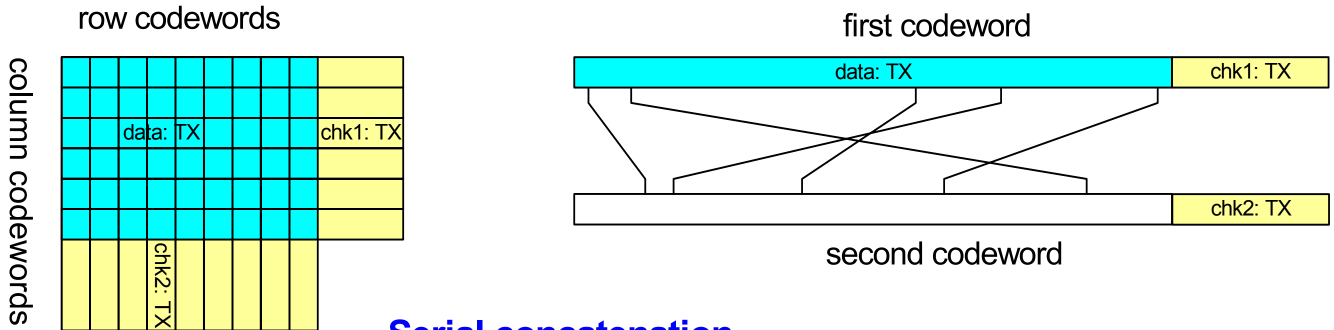
Dense lattices



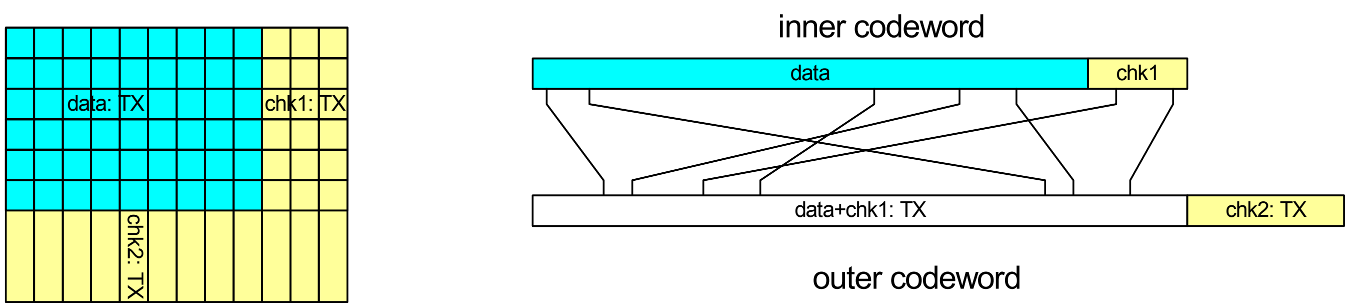
$$\text{Symbol error probability per 2-D} \approx K_{min}/(N/2) \times Q\left(\sqrt{\gamma_c \ \text{snr}}\right)$$

Types of code concatenation

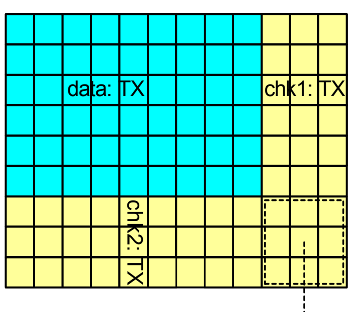
Parallel concatenation



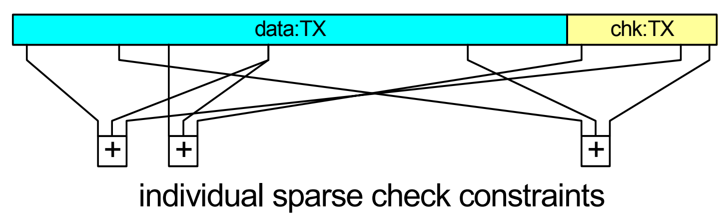
Serial concatenation



Product codes

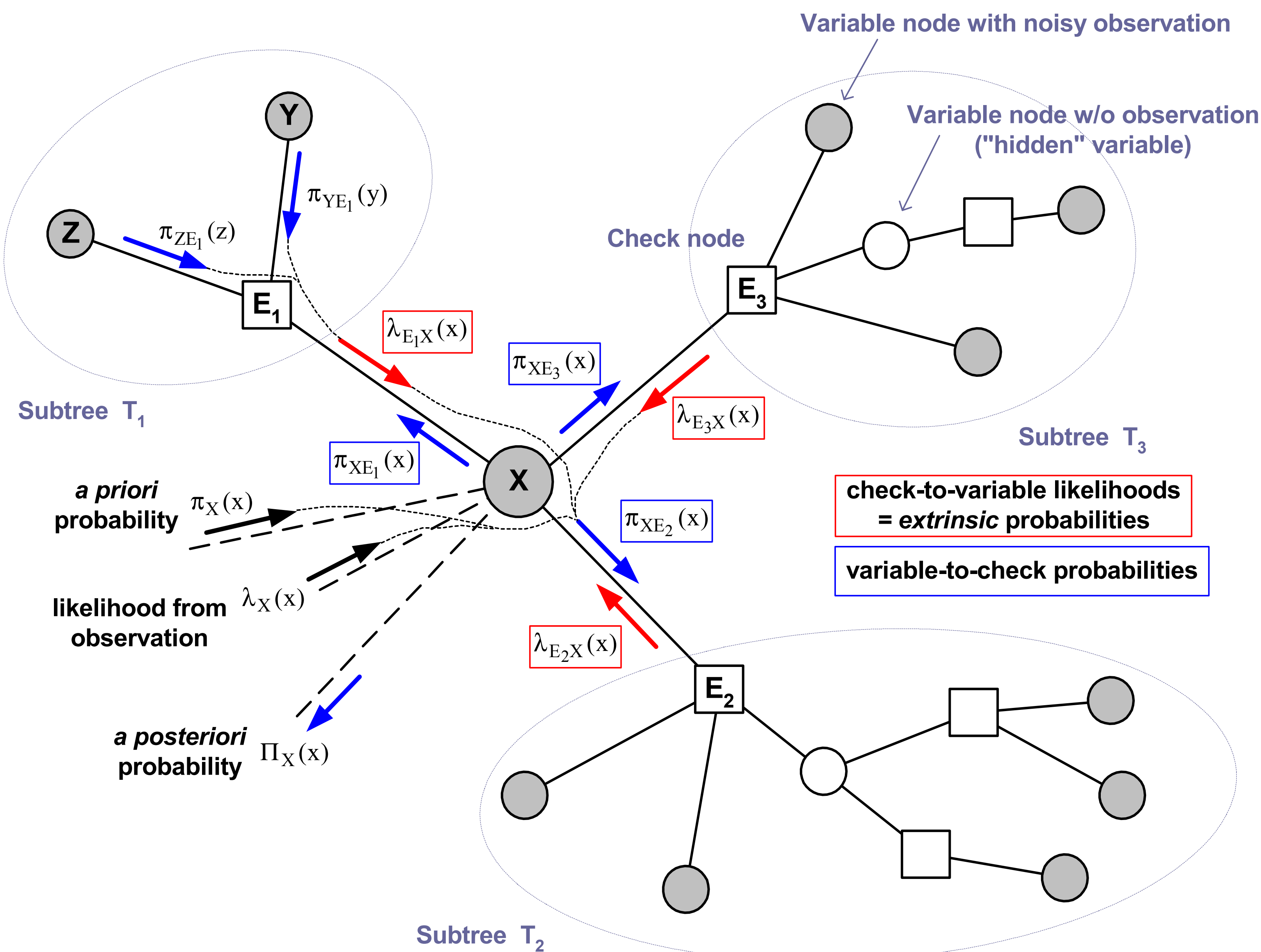


Low density parity check codes (Gallager 1963)

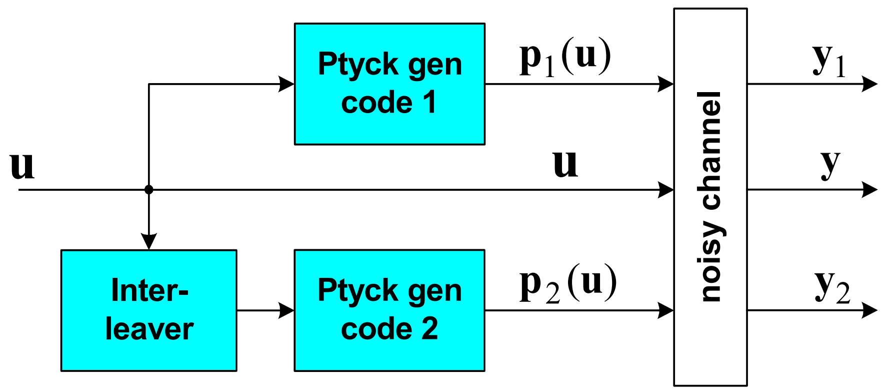


"**Checks on checks**": row and column codes must be linear and over the same field

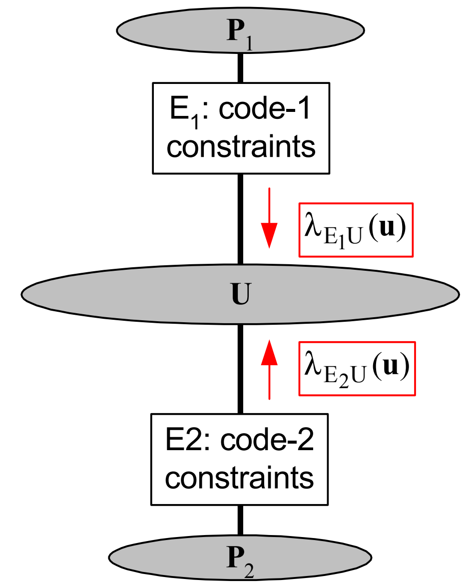
Cycle-free Tanner graph and Pearl's "belief" propagation



Parallel concatenated codes, turbo decoding (1993)



Tanner graph for vector variables

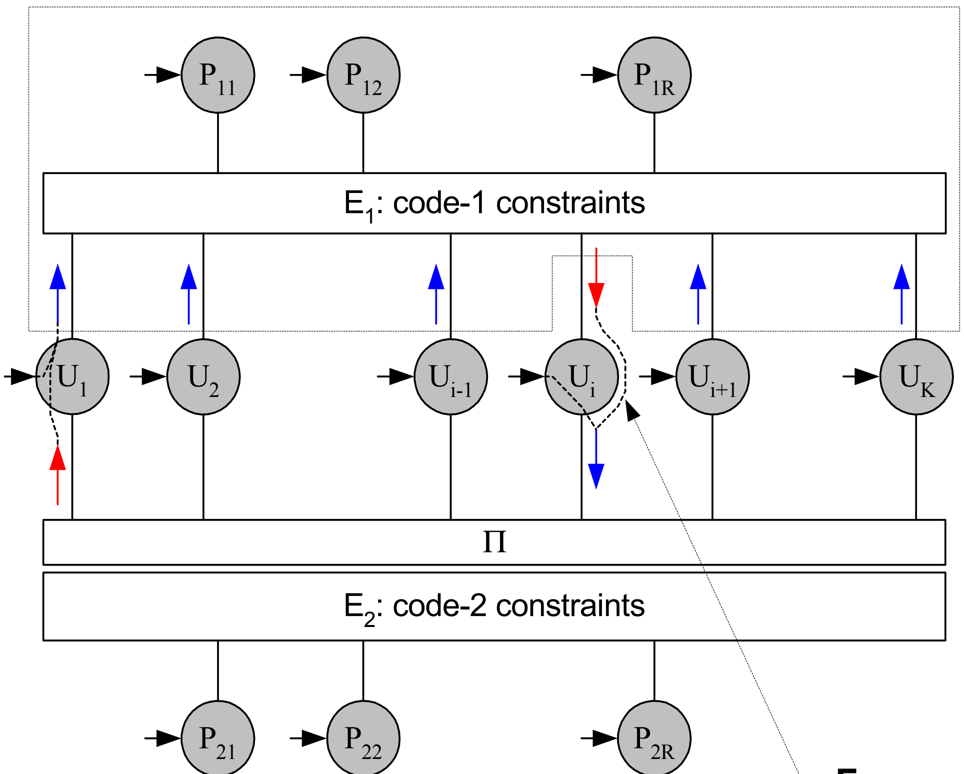


Iterative ("Turbo") decoding introduced by Berrou & Glavieux achieves near-optimum decoding by message passing of marginal probabilities. This leads to cycles in the Tanner graph. The effect of cycles is mitigated by using long random-like interleaving.

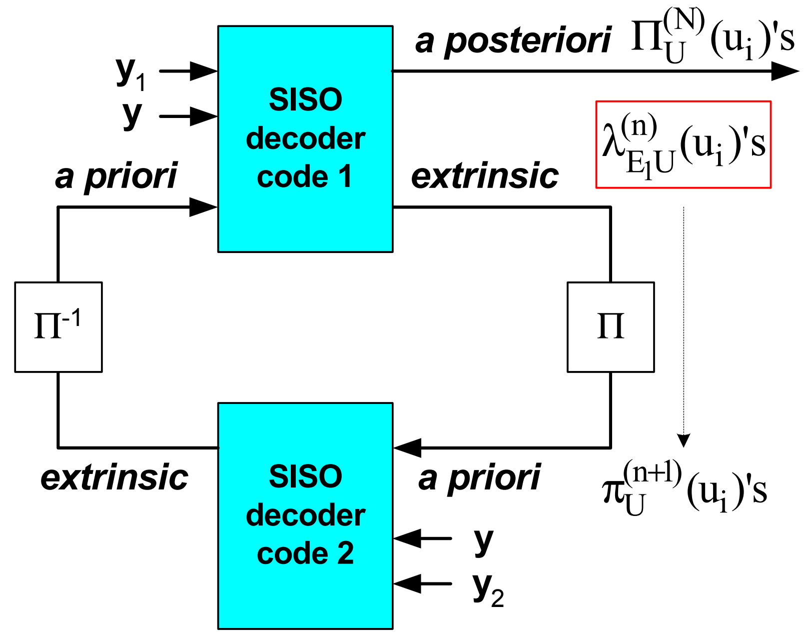
- optimum decoding (no cycles)
- requires passing of joint probabilities
- too complex, except for very short codes

Iterative decoding of parallel concatenated code (PCC)

Tanner graph explaining Turbo decoding



$n = 1, 3, \cdots N$

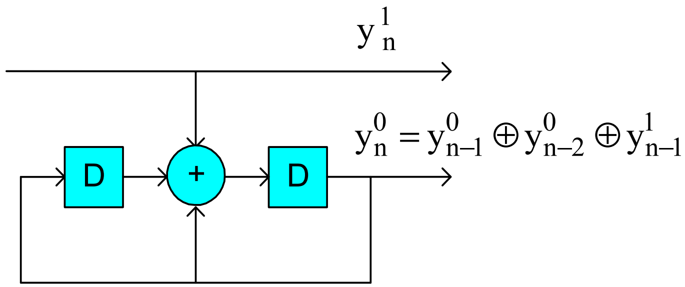


For every i (1 ≤ i ≤ K):

***extrinsic* marginal probability from decoder 1 is used as *a priori* probability for decoder 2**

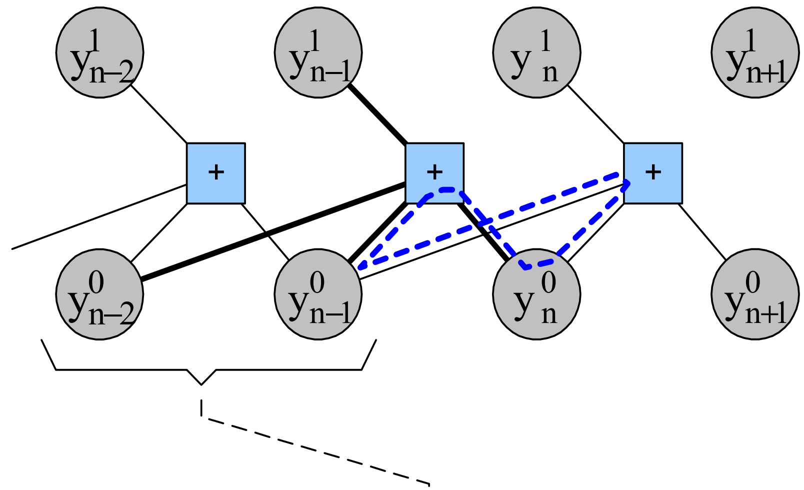
Cycles, states, and forward-backward algorithm

Recursive Systematic enCoder (SRC)

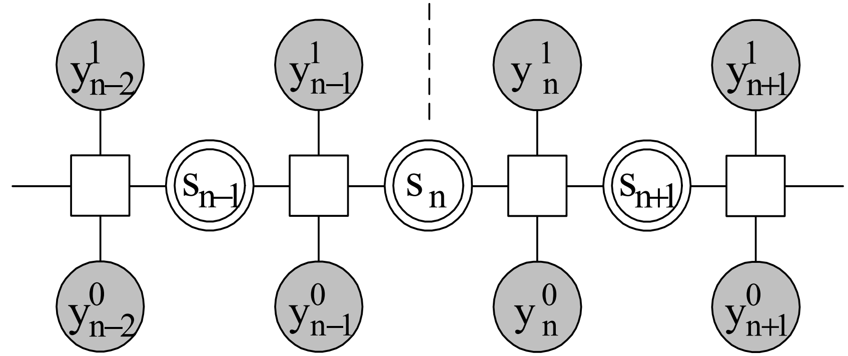


1. **Cycles can be eliminated by introducing states.**
2. **States are composite hidden variables or functions of variables.**

Tanner graph without states: cycles



Tanner-Wiberg-Loeliger graph: states eliminate cycles



Belief propagation = forward-backward = BCJR algorithm

Analysis of concatenated codes & iterative decoding

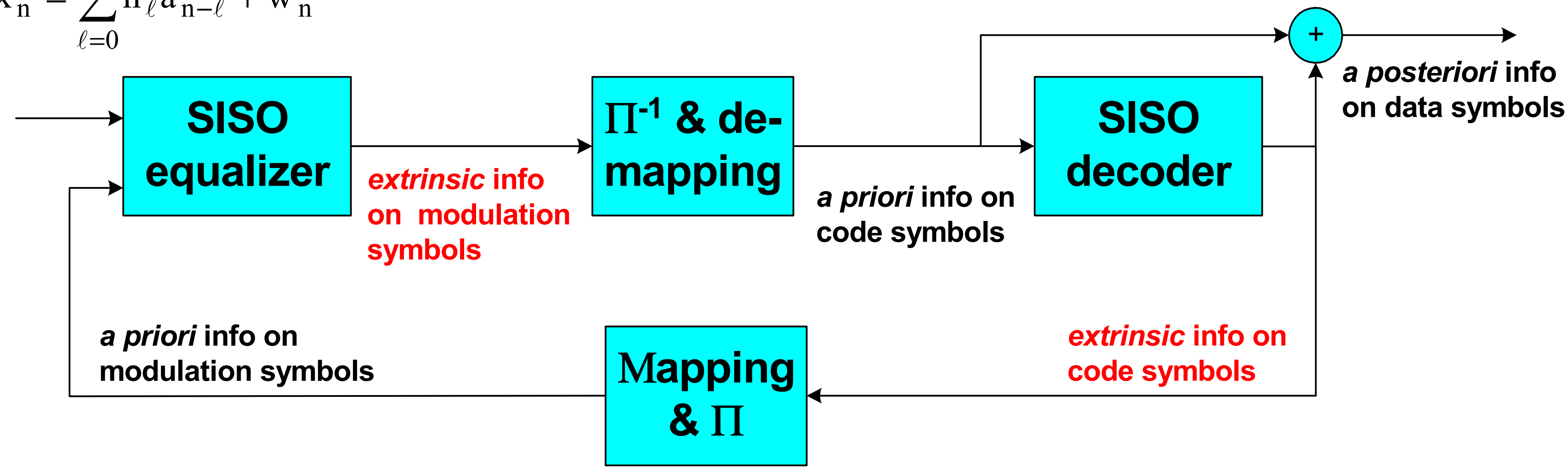
- **Benedetto, Divsalar, Montorsi, Pollara** (1996-98): estimated BER for *ML decoding* of PCCCs and SCCCs based on *average weight* enumeration assuming *uniform random interleaving.* Results illustrate interleaver gain, *near Shannon-limit performance* at moderate SNR, *change of slope* due to *low-weight codewords* at high SNR*.*

- **Gallager** (1963), **Richardson and Urbanke** (2001): *density evolution* in message-passing decoding for ensemble of random infinite-length LDPC codes; *SNR threshold for convergence* interpreted as "capacity" of LDPC code.

- **Divsalar, Hagenauer, ten Brink, et al.** (1998-2001)**:** soft-in soft-out (SISO) decoders viewed as "SNR-improving" devices $\rightarrow$ *EXIT charts*.

Iterative equalization and decoding

$$x_n = \sum_{\ell=0}^{L} h_\ell a_{n-\ell} + w_n$$

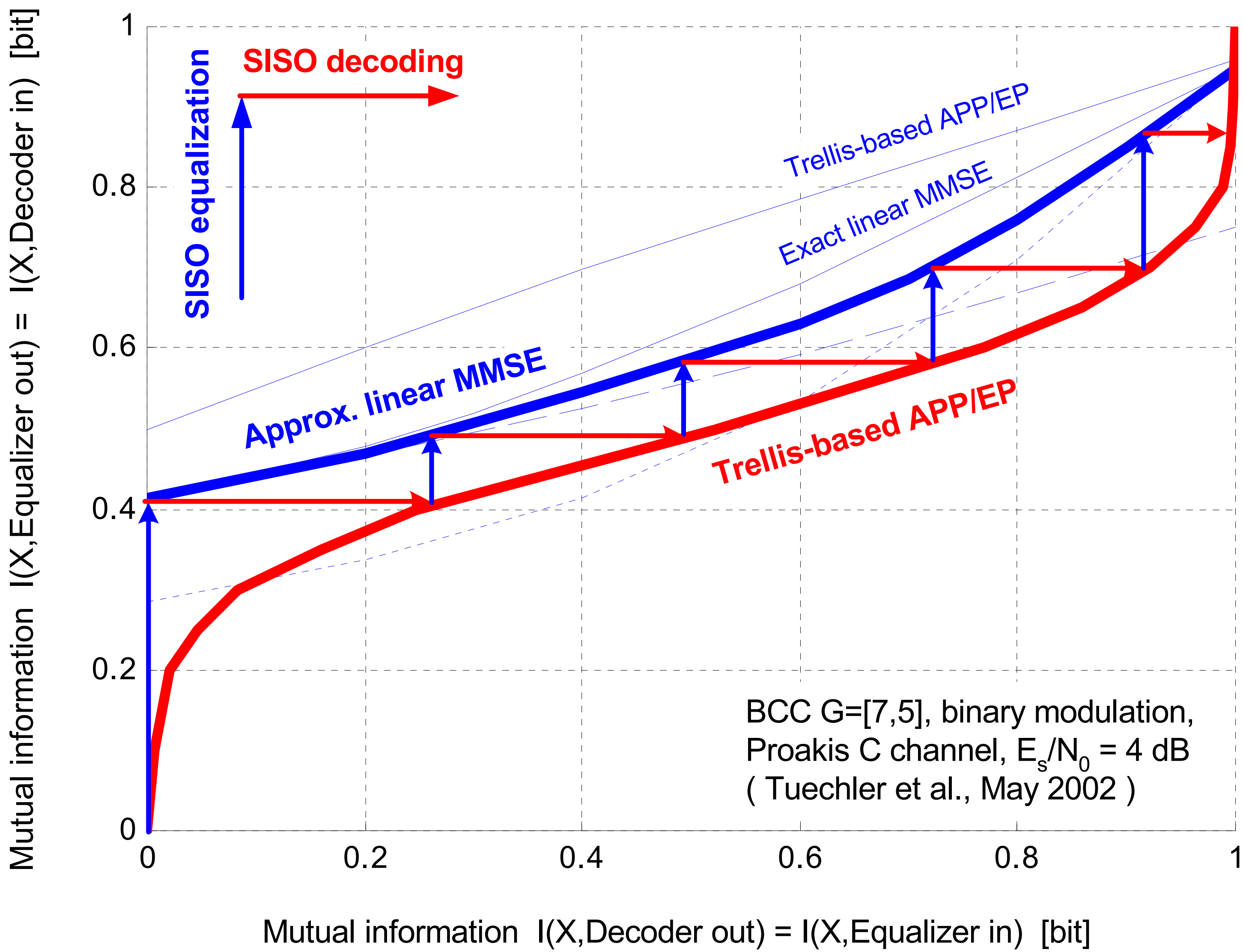


- **Trellis based APP equalizer - too complex** (except in very simple cases)
- **Linear MMSE equalizer <u>with a priori info</u> (symbol means and variances)**
  - no a priori info: ordinary MMSE equalizer

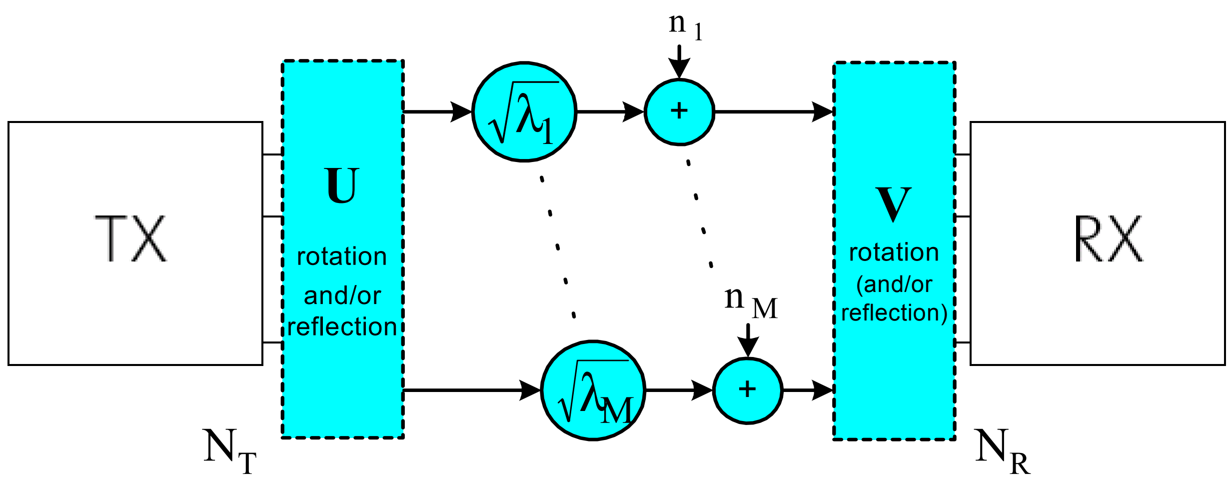
    ↓
  - perfect a priori info: ISI cancellation + matched filtering

M. Tüchler, Ralf Koetter and A. Singer, “Turbo equalization: principles and new results,“ *IEEE Trans. Comm.*, vol. 50, pp. 754-683, May 2002.
D. Raphaeli and A. Saguy, “Reduced complexity APP for turbo equalization,” IEEE Int’l Conference on Communications (ICC), vol. 3. pp. 1940-1943, 2002.

Iterative equalization and decoding: EXtrinsic Information Transfer chart



Multiple-input multiple-output (MIMO) channels



Singular value decomposition $\mathbf{H} = \mathbf{U}_{N_T} \begin{bmatrix} \sqrt{\lambda_1} & 0 & \cdots \\ 0 & \sqrt{\lambda_2} & \\ \vdots & & \ddots \\ & & \end{bmatrix} \mathbf{V}_{N_R}$

Rank $M \le \mathrm{Min}(N_T, N_R)$

Fixed $\mathbf{H}$: Capacity $C(\mathbf{H}) = \sum_{i=1}^{M} \log_2(1 + \rho\lambda_i) \; \left(= \log_2\left[\det\left(\mathbf{I} + \rho\mathbf{H}\mathbf{H}^*\right)\right]\right), \quad \rho = \frac{E_s / N_T}{N_0}$

Random $\mathbf{H}$: Ergodic capacity $\overline{C} = \underset{\mathbf{H}}{\mathrm{average}}\, C(\mathbf{H})$; Outage capacity $C_p : \Pr(C(\mathbf{H}) < C_p) = p\,(\%)$

Equalization strategies for MIMO channels

Frequency selective channels can be converted into non-frequency selective vector channels by FDM (OFDM/DMT, filter banks)

Spatial MIMO equalization

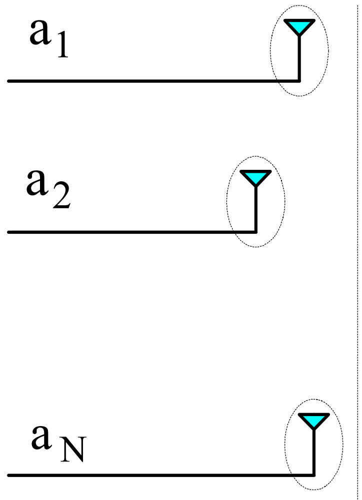
Linear at receiver	right-multiply H by $H^{-1}_{(r)}$
Linear at transmitter	left-multiply H by $H^{-1}_{(l)}$
Linear at transmitter and receiver	left-multiply H by U*, right-multiply H by V* (SVD)
Nonlinear at receiver	matrix DFE (BLAST, MUD)
Nonlinear at transmitter	matrix precoding (MUP)

(for spatial signals in row-vector form)

MIMO system: decentralized transmitters, centralized receiver

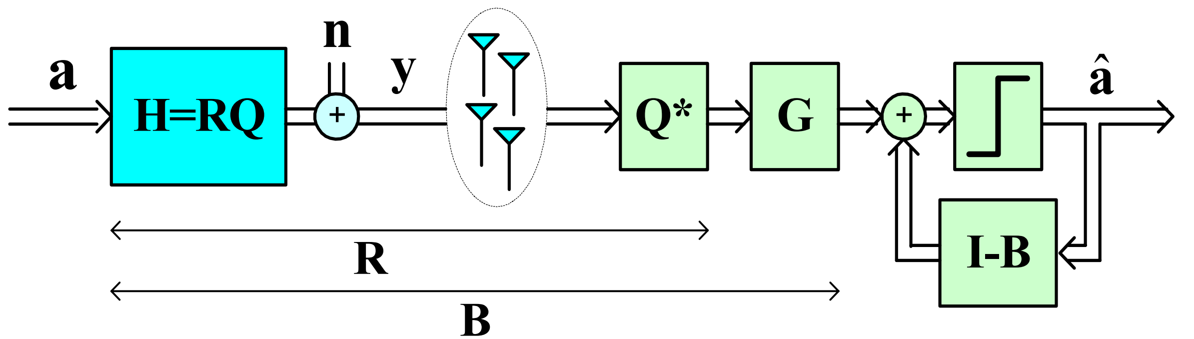
Spatial DFE = simple* multi-user detection (MUD) = BLAST

* advanced MUD employs iterative processing



$$\mathbf{a\,H + n = y}$$

Performance depends on order of detection



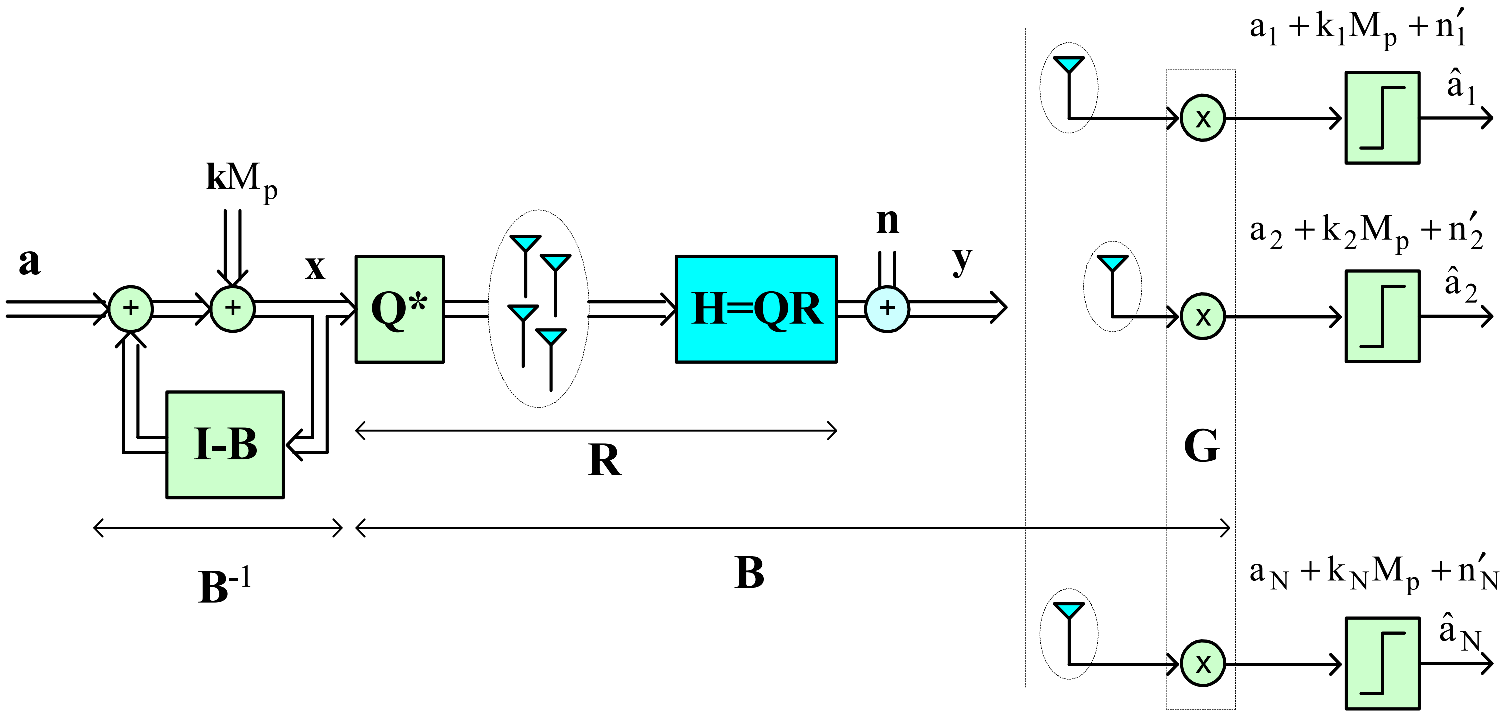
$$\underset{\mathbf{R}}{\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{bmatrix}} \underset{\mathbf{G}}{\begin{bmatrix} r_{11}^{-1} & 0 & 0 \\ 0 & r_{22}^{-1} & 0 \\ 0 & 0 & r_{33}^{-1} \end{bmatrix}} = \underset{\mathbf{B}}{\begin{bmatrix} 1 & b_{12} & b_{13} \\ 0 & 1 & b_{23} \\ 0 & 0 & 1 \end{bmatrix}}$$

$$[a_1 \quad a_2 \quad a_3]\,\mathbf{B} = [a_1 \quad a_1 b_{12} + a_2 \quad a_1 b_{13} + a_2 b_{23} + a_3]$$

QR decompositions $\mathbf{H} = \mathbf{R}_1\mathbf{Q}_1 = \mathbf{Q}_2\mathbf{R}_2 \ \ldots$ $\mathbf{R}$ = upper or lower triangular, $\mathbf{Q}$ = unitary; different orderings ...

MIMO system: central transmitter, decentralized receivers

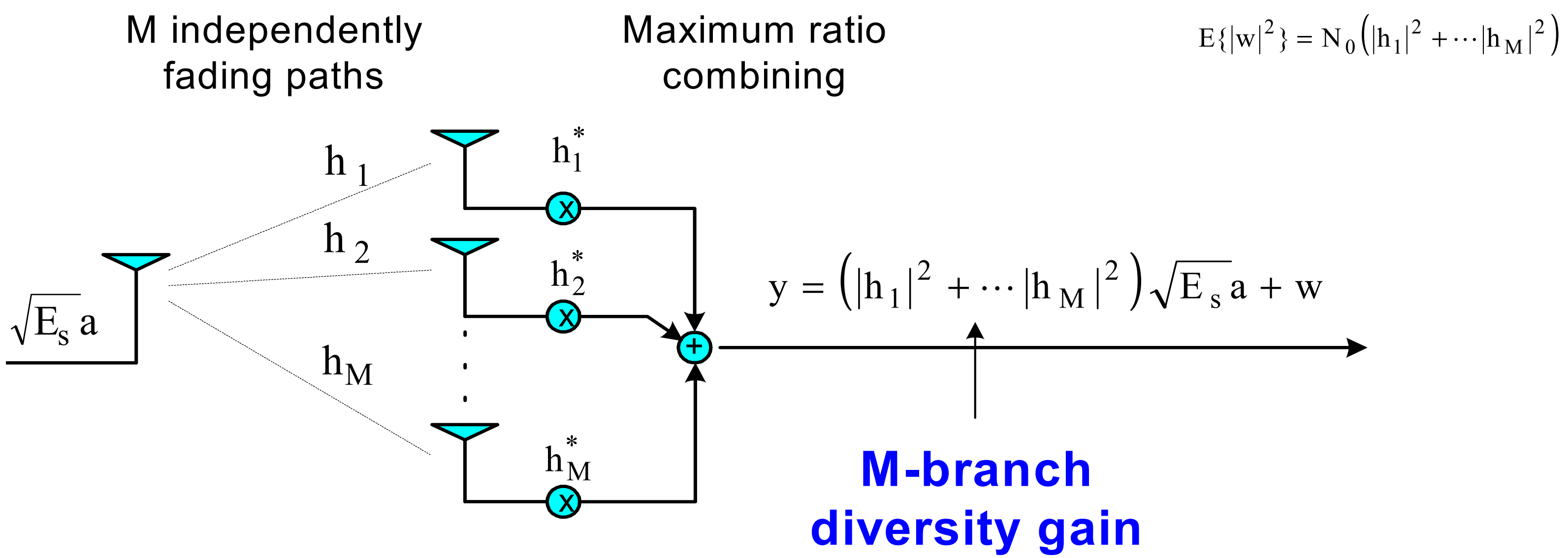
Spatial TH precoding = Multi-user precoding (MUP)

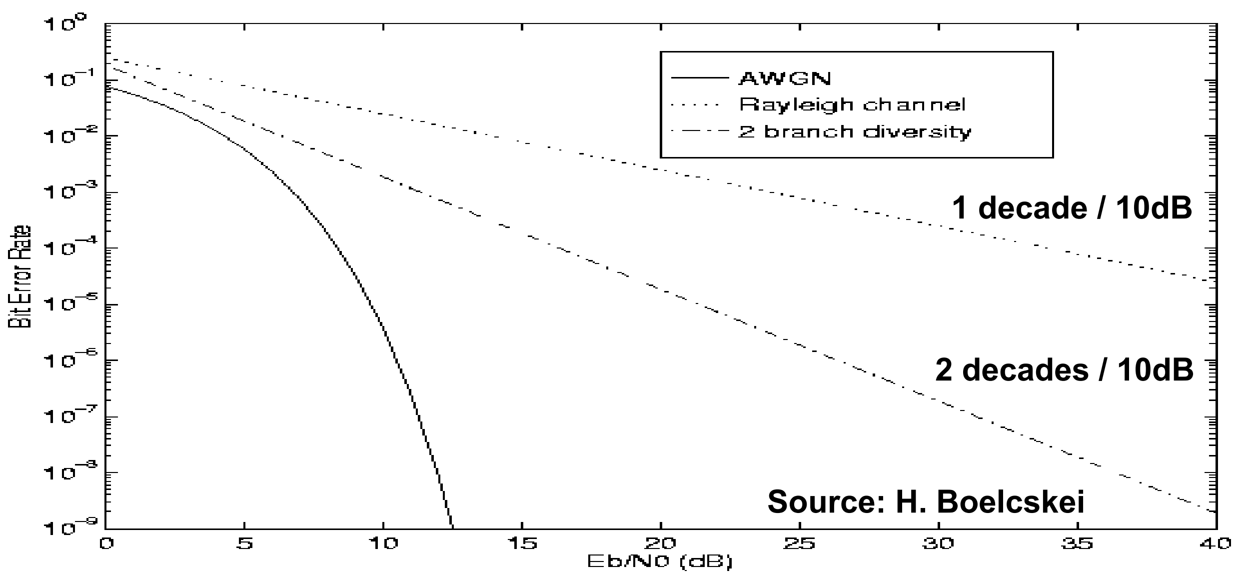


$$\left[\; x_1 = a_1 \quad x_2 = (a_2 - x_1 b_{12}) + k_2 M_p \quad x_3 = (a_3 - x_1 b_{13} - x_2 b_{23}) + k_3 M_p \;\right]$$

R. Fischer, J. Huber, and G. Komp, "Coordinated digital transmission: theory and examples," Archiv für Electronik und Übertragungstechnik, 48, pp. 289-300, Nov/Dec 1994. (see also J. Huber and R. Fischer, "Dynamically coordinated reception of multiple signals in correlated noise," Proc. of the IEEE Int'l Symp. Inf. Theory, pp.132, Trondheim, Norway, June 1994).

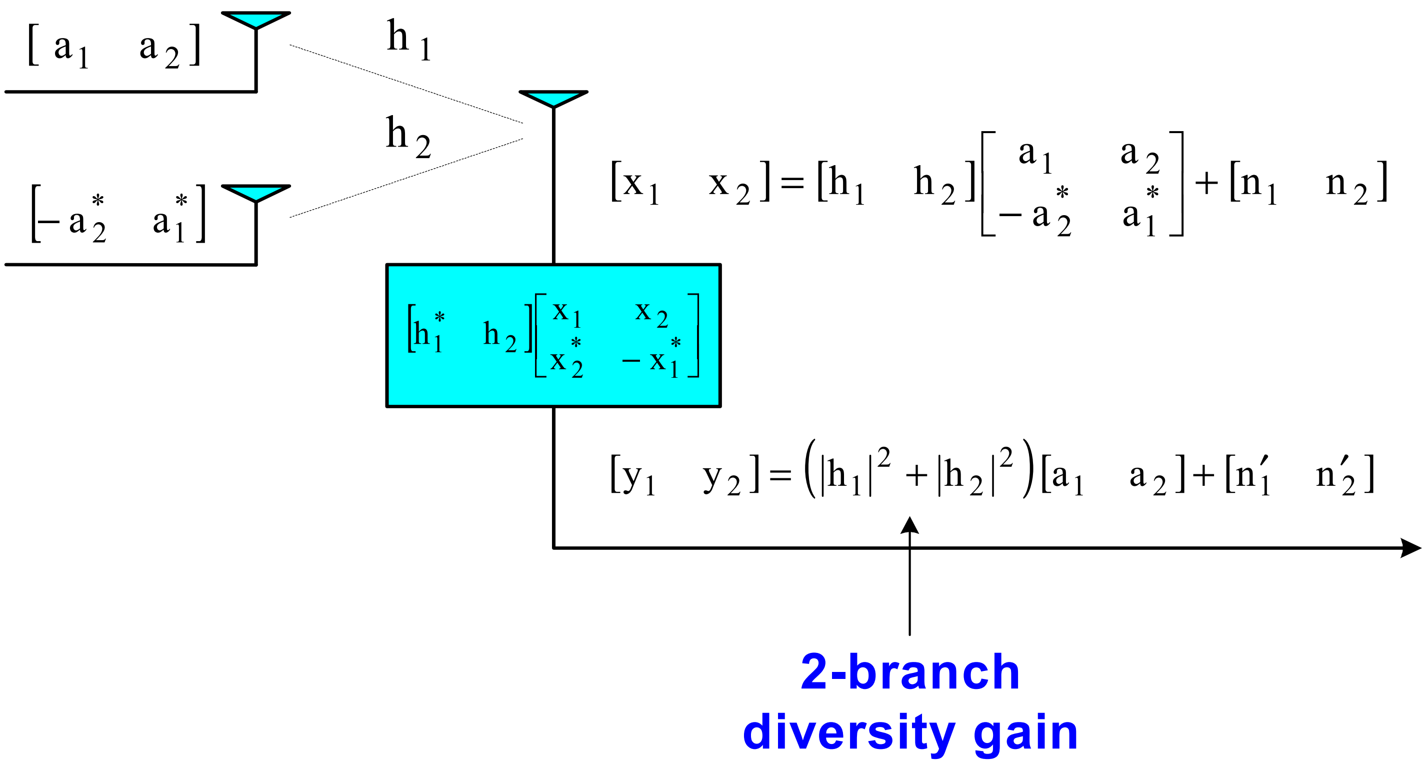
Receiver diversity





Transmit diversity and a simple space-time code

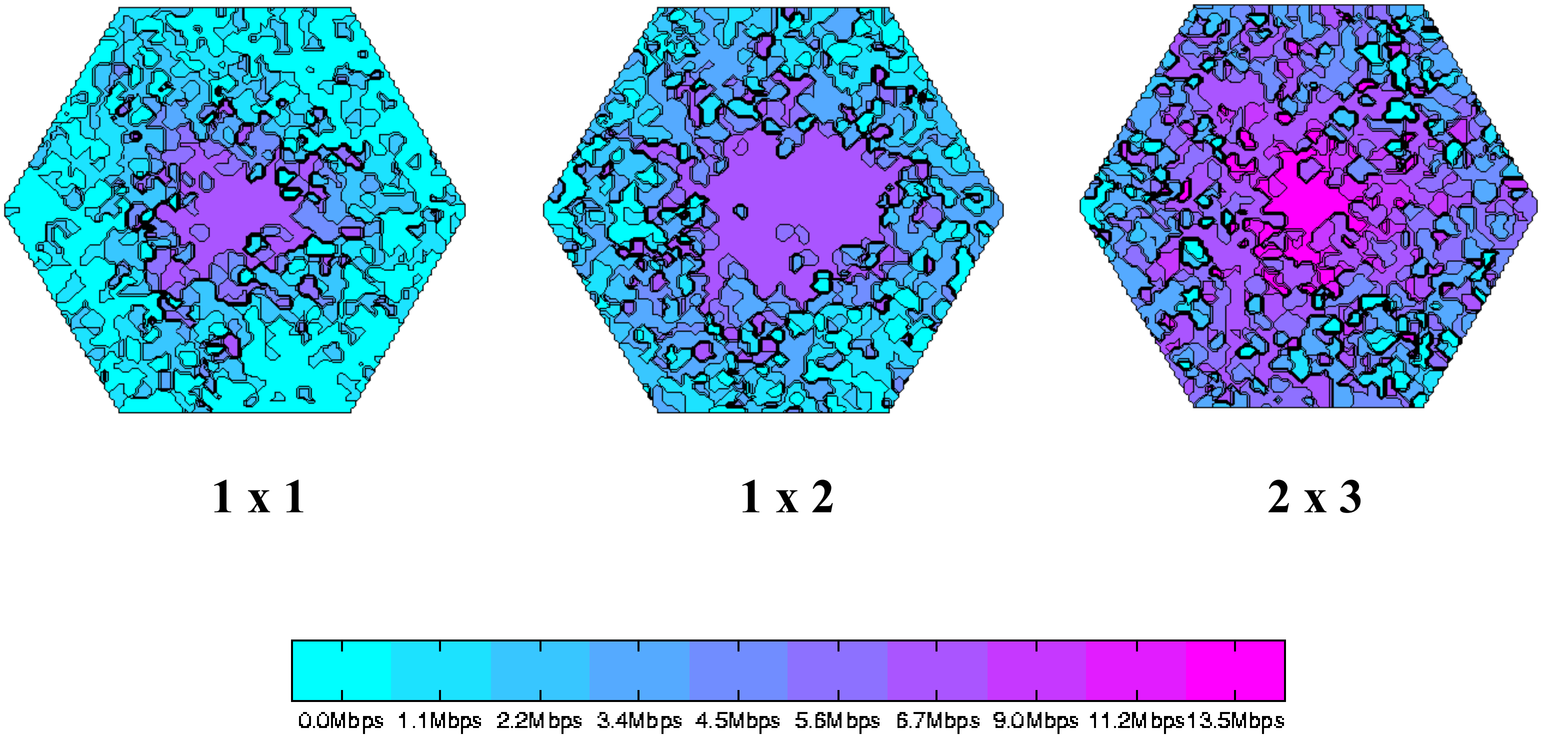
Alamouti (IEEE JSAC, Nov 98)



MIMO channels and space-time coding: objectives

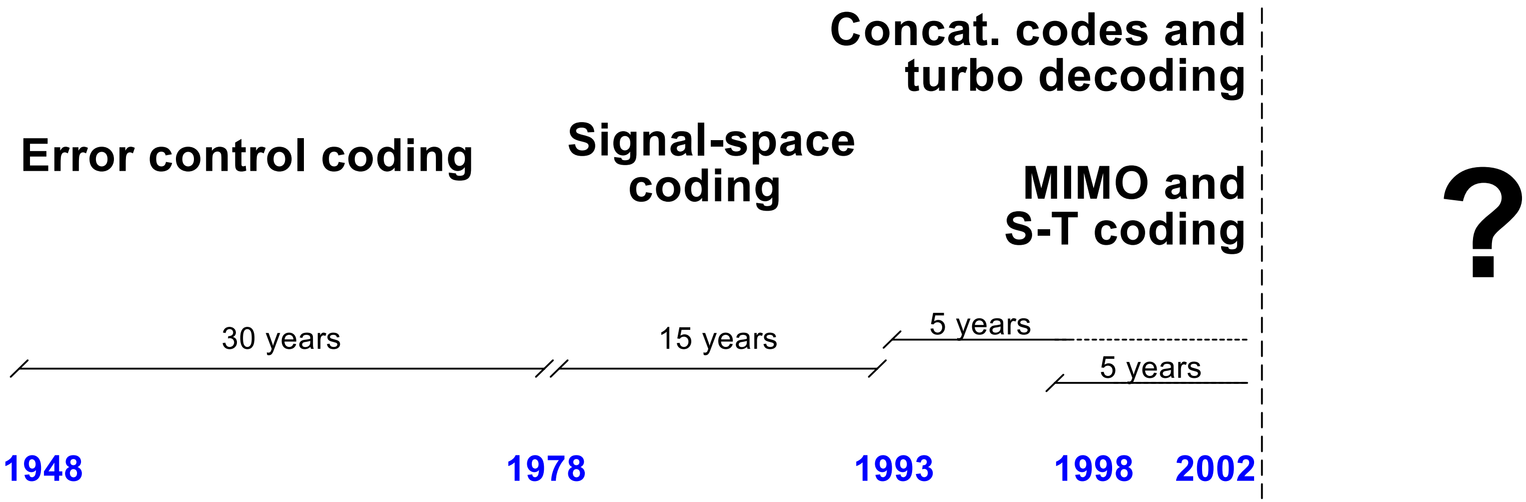
- **Spatial multiplexing gain**
  - creation of multiple channels within same bandwidth
- **Diversity gain**
  - mitigation of fading losses by averaging over individually fading paths
- **Coding gain**
  - increasing distance between codewords / sequences by improved arrangements of points in higher dimensional spaces

Throughput in MIMO-OFDM cellular systems



Source: H. Boelcskei

What comes next in coding?



(a) That's all, folks!

(b) Another big topic?

(c) Extended phase of consolidations?

Challenges ahead

- **Develop new standards exploiting the exciting potential of MIMO and S/T coding:** G3+ cellular systems, Gbit/s wireless Ethernet, … using base stations with multiple antennas, mobile stations with 1 - 2 antennas.
  - optimize network capacity by S/T coding, multi-user detection, multi-user precoding, beam forming, …
  - channel estimation; PHY and MAC protocols
- **Performance / complexity in turbo and S/T coding:** many schemes proposed, few adopted in applications, what is most useful?

Challenges ahead

- **Concatenated codes & iterative decoding for very low error rates:** e.g., 10 Gbit/s optical links, BER=10^{-12} – 10^{-15}.

- **Soft-decoding of widely used algebraic FEC codes:** known soft-decoding algorithms are too complex or provide only small gain, for e.g., RS(255,239).

:

- **Designing chips of massive complexity**

- **Develop market for these new technologies. Succeed under slower market conditions.**